7

Kinetic Theory

So far we have concentrated on collisionless systems, in which the constituent particles move under the influence of the smoothed-out gravitational field generated by all the other particles. This approximation is not completely accurate. As described in §1.2, individual stellar encounters[1] gradually perturb stars away from the trajectories they would have taken if the gravitational field were perfectly smooth; in effect the stars diffuse in phase space away from their original orbits. After many such encounters the star eventually loses its memory of the original orbit, and finds itself on a wholly unrelated one. The characteristic time over which this loss of memory occurs is called the relaxation time $t_{\rm relax}$; over timescales exceeding $t_{\rm relax}$ the approximation of a smooth gravitational potential is incorrect.

The collisionless Boltzmann equation, which has been our main tool so far, is not valid when encounters are important. Thus we begin this chapter by reviewing general results about stellar systems that hold in the presence of encounters (§7.2 and §7.3). The equations that describe the behavior of stellar systems in the presence of encounters are derived in §7.4, and these are used to investigate the evolution of spherical stellar systems in §7.5.

[1] We generally use the term "encounter" to denote the gravitational perturbation of the orbit of one star by another, and "collision" to denote actual physical contact between stars. However, to conform with common use, we use the terms "collisional" or "collisionless" to describe stellar systems in which encounters do or do not play a role.

For other discussions of the topics in this chapter, see Hénon (1973b), Spitzer (1987), and Heggie & Hut (2003).

7.1 Relaxation processes

The relaxation time is of order

$$t_{\rm relax} \approx \frac{0.1N}{\ln N} t_{\rm cross}, \tag{7.1}$$

where $t_{\rm cross}$ is the crossing time and N is the number of stars in the system (eq. 1.38). The relaxation time exceeds the crossing time if $N \gtrsim 40$. Galaxies typically have $N \approx 10^{11}$ and $t_{\rm cross} \approx 100\,{\rm Myr}$, so the effects of stellar encounters can be ignored over a galaxy's lifetime of 10 Gyr. However, encounters have played a central role in determining the present structure of many other stellar systems, such as globular clusters ($N \approx 10^5$, $t_{\rm cross} \approx 10^5\,{\rm yr}$, lifetime 10 Gyr), open clusters ($N \approx 10^2$, $t_{\rm cross} \approx 1\,{\rm Myr}$, lifetime 100 Myr), the central parsec of galaxies ($N \approx 10^6$, $t_{\rm cross} \approx 10^4\,{\rm yr}$, lifetime 10 Gyr), and the centers of clusters of galaxies ($N \approx 10^3$, $t_{\rm cross} \approx 1\,{\rm Gyr}$, lifetime 10 Gyr).

The fundamental equations describing motion in a collisionless system of N stars of mass m are the collisionless Boltzmann and Poisson equations (eqs. 4.7 and 2.10),

$$\frac{\partial f}{\partial t} + [f, H] = 0 \quad ; \quad \nabla^2 \Phi(\mathbf{x}, t) = 4\pi G m N \int \mathrm{d}^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t). \tag{7.2}$$

Here the Hamiltonian $H(\mathbf{x}, \mathbf{v}, t) = \frac{1}{2}v^2 + \Phi(\mathbf{x}, \mathbf{v}, t)$ and the DF $f(\mathbf{x}, \mathbf{v}, t)$ represents the probability that a given star is found in unit phase-space volume near the phase-space position $(\mathbf{x}, \mathbf{v})$. In Chapter 4 we developed models of stellar systems by solving these equations exactly. For example, in spherical models such as the Hernquist model, the gravitational field is precisely time-independent and spherical, so each star conserves its energy and angular momentum. However, in any stellar system with finite N, the energy and angular momentum of individual stars are not precisely conserved, because each star is subject to fluctuating forces from encounters with its neighbors. Therefore the collisionless Boltzmann equation does not provide a complete description of the dynamics of stellar systems with finite N.

Encounters drive the evolution of a stellar system by several distinct mechanisms:

(a) Relaxation Each star slowly wanders away from its initial orbit. As a result of this phase-space diffusion, the entropy of the stellar system increases, and its structure becomes less sensitive to its initial conditions. We have seen in §4.10.1 that the high-entropy states of a self-gravitating gas are very inhomogeneous, with a dense central core and an extended halo. Thus we expect that relaxation will drive stellar systems towards configurations having small, dense cores and large, low-density halos.

(b) Equipartition A typical stellar system contains stars with a wide range of masses. From elementary kinetic theory we know that encounters tend to produce equipartition of kinetic energy: on average, particles with large kinetic energy $\frac{1}{2}mv^2$ lose energy to particles with less kinetic energy. In an ordinary gas, this process leads to a state in which the mean-square velocity of a population of particles is inversely proportional to mass. By contrast, in a stellar system, massive stars that lose kinetic energy fall deeper into the gravitational potential well, and may even increase their kinetic energy as a result, just as an Earth satellite speeds up as it loses energy from atmospheric drag. Conversely, less massive stars preferentially diffuse towards the outer parts of the stellar system, where the velocity dispersion may be smaller.

(c) Escape From time to time an encounter gives a star enough energy to escape from the stellar system. Thus there is a slow but irreversible leakage of stars from the system, so stellar systems gradually evolve towards a final state consisting of only two stars in a Keplerian orbit, all the others having escaped to infinity. The timescale over which the stars "evaporate" in this way can be related to the relaxation timescale by the following simple argument (Ambarzumian 1938; Spitzer 1940). From equation (2.31) the escape speed v_e at $\mathbf{x}$ is given by $v_\mathrm{e}^2(\mathbf{x}) = -2\Phi(\mathbf{x})$. The mean-square escape speed in a system whose density is $\rho(\mathbf{x})$ is therefore

$$\langle v_\mathrm{e}^2\rangle = \frac{\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})v_\mathrm{e}^2(\mathbf{x})}{\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})} = -2\frac{\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})\Phi(\mathbf{x})}{M} = -\frac{4W}{M}, \tag{7.3}$$

where M and W are the total mass and potential energy of the system (eq. 2.18). According to the virial theorem (4.250), $-W = 2K$, where $K = \frac{1}{2}M\langle v^2\rangle$ is the total kinetic energy. Hence

$$\langle v_\mathrm{e}^2\rangle^{1/2} = 2\langle v^2\rangle^{1/2}; \tag{7.4}$$

in words, the RMS escape speed is just twice the RMS speed. The fraction of particles in a Maxwellian distribution that have speeds exceeding twice the RMS speed is $\gamma = 7.38\times 10^{-3}$ (Problem 4.18). We can crudely assume that evaporation removes a fraction γ of the stars every relaxation time. Then the rate of loss is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{\gamma N}{t_\mathrm{relax}} \equiv -\frac{N}{t_\mathrm{evap}}, \tag{7.5}$$

where the **evaporation time**, the characteristic time in which the system's stars are lost, is $t_\mathrm{evap} = t_\mathrm{relax}/\gamma \simeq 140\,t_\mathrm{relax}$. Thus we expect that any stellar system will lose a substantial fraction of its stars in about $10^2 t_{\rm relax}$ (see §7.5.2).

(d) Inelastic encounters So far we have treated stars as point masses, but in dense stellar systems we must consider the possibility that two stars occasionally pass so close that they raise powerful tides on one another or even suffer a physical collision. Energy dissipation in near-collisions reduces the total kinetic energy of the system and can lead to the formation of binary stars. Head-on or nearly head-on collisions can result in the coalescence of the colliding stars, leading to the otherwise unexpected presence of massive, short-lived stars in an old stellar system.

The characteristic timescale on which a star suffers a collision is given approximately by

$$t_{\rm coll} \approx (n\Sigma v)^{-1}, \tag{7.6}$$

where n is the number density of stars, Σ is the collision cross-section, and v is the RMS stellar velocity. We may write $n \approx N/r^3$, where r is the radius of the system, and $\Sigma \approx \pi(2r_\star)^2$, where $r_\star$ is the stellar radius (neglecting gravitational focusing; a more precise result is given in eq. 7.194). In terms of the crossing time $t_{\rm cross} \approx r/v$,

$$\frac{t_{\rm coll}}{t_{\rm cross}} \approx \frac{r^2}{4\pi N r_\star^2}. \tag{7.7}$$

From the virial theorem we have $v^2 \approx GNm/r$ where m is the stellar mass; it proves convenient to use this relation to eliminate r in favor of v. We also eliminate $r_\star$ in favor of the escape speed from the stellar surface, $v_\star = \sqrt{2Gm/r_\star}$ ($v_\star = 618\,{\rm km\,s^{-1}}$ for the Sun). Then

$$\frac{t_{\rm coll}}{t_{\rm cross}} \approx 0.02N\left(\frac{v_\star}{v}\right)^4. \tag{7.8}$$

In terms of the relaxation time (eq. 7.1),

$$\frac{t_{\rm coll}}{t_{\rm relax}} \approx 0.2\left(\frac{v_\star}{v}\right)^4 \ln N. \tag{7.9}$$

For systems in which the escape speed from individual objects is much larger than the RMS orbital velocity (such as open and globular clusters, and most galaxies), we have $t_{\rm coll} \gg t_{\rm relax}$, so inelastic encounters play only a minor role in determining the overall structure of the stellar system. However, such encounters can occasionally produce exotic single or binary stars, which provide direct evidence of recent non-gravitational interactions.

(e) Binary formation by triple encounters A binary star cannot form in an isolated encounter of two point masses, because the relative motion is always along a hyperbola. However, an encounter involving three stars can leave two of the participants in a bound Keplerian orbit. It is simple to estimate the rate of formation of binaries by this process. We showed in equation (1.30) that the velocity perturbation in an encounter of two stars of mass m and relative velocity v is $\delta v \approx Gm/bv$, where b is the distance of closest approach. We may rewrite this as

$$\frac{\delta v}{v} \approx \frac{b_{90}}{b}, \quad \text{where} \quad b_{90} \approx \frac{Gm}{v^2} \tag{7.10}$$

is the impact parameter at which the relative velocity is deflected by 90° in the encounter (see eq. 3.51 for a precise definition). If three stars approach one another within a distance b, we expect the velocity perturbations to be of similar magnitude. Thus, to form a binary by a triple encounter, we must have $\delta v \approx v$, which requires $b \approx b_{90}$. For a given star, the time interval between encounters with other stars at separation b_{90} or less is of order $(nb_{90}^2 v)^{-1}$ (eq. 7.6). In each such encounter, there is a probability nb_{90}^3 that a third star will also lie within a distance b_{90}. Hence the time t_3 required for a given star to suffer a triple encounter at separation less than b_{90} is $t_3 \approx (n^2 b_{90}^5 v)^{-1}$. Substituting for b_{90} from equation (7.10), we find the time required for a given star to become part of a binary by a triple encounter to be (Goodman & Hut 1993)

$$t_3 \approx \frac{v^9}{n^2 G^5 m^5}. \tag{7.11}$$

Using the virial theorem, $v^2 \approx GNm/r$, we may express t_3 in terms of the relaxation time (eq. 7.1):

$$\frac{t_3}{t_{\rm relax}} \approx 10N^2 \ln N. \tag{7.12}$$

Hence the total number of binaries formed per relaxation time is only

$$\frac{Nt_{\rm relax}}{t_3} \approx \frac{0.1}{N \ln N}. \tag{7.13}$$

Since the system dissolves after the evaporation time of about $10^2 t_{\rm relax}$, the rate of binary formation by triple encounters is negligible if N is much larger than 10. We discuss binary formation and evolution further in §7.5.7.

(f) Interactions with primordial binaries The many binary stars found in the solar neighborhood were produced when their component stars were formed, rather than by subsequent triple or inelastic encounters. It is likely that binary stars are similarly produced during the formation of globular and open clusters. These are called **primordial binary stars** to distinguish them from binaries formed by dynamical processes long after their constituent stars. Gravitational forces during encounters transfer energy between the orbits of primordial binaries and other cluster stars. Such energy exchange can dramatically alter the energy balance in the cluster, even if binaries are rare, because the binding energy in the binary orbit can be much larger than the kinetic energy of a typical cluster star. Consider, for example, a globular cluster with mass $M = 10^5\,\mathcal{M}_\odot$ and RMS velocity $\langle v^2\rangle^{1/2} = 10\,\mathrm{km\,s^{-1}}$. From the virial theorem, its binding energy is $-\frac{1}{2}M\langle v^2\rangle = 10^{50}$ erg. A binary star consisting of two $1\,\mathcal{M}_\odot$ stars with a separation of $2\,R_\odot$ has a binding energy of 1×10^{48} erg. Thus, 100 such binaries contain as much binding energy as the whole cluster of 10^5 stars.

7.2 General results

7.2.1 Virial theorem

In Chapter 4 we used the collisionless Boltzmann equation to prove the tensor virial theorem (eqs. 4.241 and 4.247),

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2K_{jk} + W_{jk}, \tag{7.14}$$

which relates the tensor I_{jk} of an isolated stellar system to the kinetic- and potential-energy tensors, K_{jk} and W_{jk}. We now show that with slight modifications this result also holds for collisional systems.

Proof: Consider a system of particles with masses m_α and positions $\mathbf{x}_\alpha, \alpha = 1,\ldots,N$. We define the tensor (cf. eq. 4.243)

$$I_{jk} \equiv \sum_{\alpha=1}^{N} m_\alpha x_{\alpha j} x_{\alpha k}, \tag{7.15}$$

where $x_{\alpha j}$ is the jth Cartesian component of the vector $\mathbf{x}_\alpha$. The second time derivative of I_{jk} is

$$\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = \sum_{\alpha=1}^{N} m_\alpha \left(x_{\alpha j}\ddot{x}_{\alpha k} + 2\dot{x}_{\alpha j}\dot{x}_{\alpha k} + \ddot{x}_{\alpha j} x_{\alpha k}\right). \tag{7.16}$$

The acceleration of particle α is

$$\ddot{x}_{\alpha j} = \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{G m_\beta \bigl(x_{\beta j} - x_{\alpha j}\bigr)}{|\mathbf{x}_\beta - \mathbf{x}_\alpha|^3}; \tag{7.17}$$

substituting this result and a similar formula for $\ddot{x}_{\alpha k}$ into equation (7.16) we find

$$\begin{aligned}\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} &= 2\sum_{\alpha=1}^{N} m_\alpha \dot{x}_{\alpha j}\dot{x}_{\alpha k} \\ &\quad + \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}\left[x_{\alpha j}\big(x_{\beta k} - x_{\alpha k}\big) + x_{\alpha k}\big(x_{\beta j} - x_{\alpha j}\big)\right].\end{aligned} \tag{7.18}$$

By analogy with equation (4.240b), we define the kinetic-energy tensor for a system of point particles to be

$$K_{jk} \equiv \tfrac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{x}_{\alpha j}\dot{x}_{\alpha k}. \tag{7.19}$$

By analogy with equations (2.21a) and (2.22), we define the potential-energy tensor for a system of point particles as[2]

$$\begin{aligned} W_{jk} &= G\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} m_\alpha m_\beta \frac{x_{\alpha j}(x_{\beta k} - x_{\alpha k})}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} \\ &= -\tfrac{1}{2}G\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} m_\alpha m_\beta \frac{(x_{\alpha j} - x_{\beta j})(x_{\alpha k} - x_{\beta k})}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3},\end{aligned} \tag{7.20}$$

where the second line is obtained by interchanging the indices α and β in the first line and averaging this result with the first line. From the second line we conclude that $\mathbf{W}$ is symmetric, that is, $W_{jk} = W_{kj}$. The second term on the right side of equation (7.18) is just $W_{jk} + W_{kj} = 2W_{jk}$, and the first term is $4K_{jk}$, so we have successfully arrived at equation (7.14).◁

The most useful form of the virial theorem is obtained by taking the trace of the tensor $\mathbf{I}$, $I \equiv \mathrm{trace}\,(\mathbf{I}) \equiv \sum_{j=1}^{3} I_{jj}$. Furthermore, we assume that the system is in a steady state, so $\mathrm{d}^2 I/\mathrm{d}t^2 = 0$. The trace of equation (7.18) then becomes the scalar virial theorem $2K + W = 0$ (eq. 4.248), where now

$$K = \mathrm{trace}\,(\mathbf{K}) = \tfrac{1}{2}\sum_{\alpha=1}^{N} m_\alpha v_\alpha^2 \quad ; \quad W = \mathrm{trace}\,(\mathbf{W}) = -\tfrac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|} \tag{7.21}$$

[2] We can justify these analogies, at least formally, by replacing the continuous density $\rho(\mathbf{x})$ in (2.21a) and (2.22) by a sum of delta functions: $\rho(\mathbf{x}) = \sum_{\alpha=1}^{N} m_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha)$.

are the total kinetic and potential energies.

The only approximation involved in deriving the scalar virial theorem is that I is time-independent. This is a good approximation for equilibrium stellar systems with $N \gg 1$, but in a system with a small number of particles there are statistical fluctuations in I, so the scalar virial theorem holds only for the time-averaged values of K and W.

7.2.2 Liouville's theorem

We have argued that the collisionless Boltzmann equation cannot provide a complete description of the dynamics of a stellar system with finite N. We now discuss a generalization of the collisionless Boltzmann equation that remedies this shortcoming, at least formally. We represent the state of a system of N stars by a point in a $6N$-dimensional space, called **Γ-space**, whose coordinates are the positions and velocities of all the stars. This state is sometimes called a **microstate** and its representative point a **Γ-point**. In practice, we do not have—and do not want—the detailed information that is required to specify a microstate. We are concerned only with the "average" behavior of the macroscopic properties of the system (density distribution, velocity distribution at a given position, fraction of binary stars, etc.). Thus it is useful to imagine that at some initial time we are given the probability that a system is found in each small volume in Γ-space, and to follow the evolution of this probability distribution, rather than the evolution of a single Γ-point. There is an obvious analogy to the methods of Chapter 4, where we found it simpler to follow the evolution of the probability density in six-dimensional phase space, rather than the orbits of individual stars.

Denote the position and velocity of the αth particle by the canonical coordinates $\mathbf{q}_\alpha, \mathbf{p}_\alpha$, where $\alpha = 1, \ldots, N$ (normally $\mathbf{q}_\alpha$ and $\mathbf{p}_\alpha$ are the position and velocity, but they could be any canonical coordinates and momenta). Then the six-dimensional vector $\mathbf{w}_\alpha \equiv (\mathbf{q}_\alpha, \mathbf{p}_\alpha)$ denotes the location of a particle in phase space. The Γ-point of a system in the $6N$-dimensional Γ-space is determined by the collection of N six-vectors $\mathbf{w}_1, \ldots, \mathbf{w}_N$. The probability that a Γ-point is found in a unit volume of Γ-space at time t is denoted by $f^{(N)}(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$; since the probability density integrates to unity we have

$$\int \mathrm{d}^6\mathbf{w}_1 \cdots \mathrm{d}^6\mathbf{w}_N \, f^{(N)}(\mathbf{w}_1, \ldots, \mathbf{w}_N, t) = 1, \quad \text{where} \quad \mathrm{d}^6\mathbf{w}_\alpha \equiv \mathrm{d}^3\mathbf{q}_\alpha \, \mathrm{d}^3\mathbf{p}_\alpha. \tag{7.22}$$

The function $f^{(N)}$ is the **N-body distribution function** or N-body DF.

For the sake of simplicity, we shall usually assume that all the particles are identical (same mass, composition, etc.)—it is straightforward to modify the derivations below when several different kinds of particle are present.

Since the particles are identical, the N-body DF can be taken to be a symmetric function of $\mathbf{w}_1, \ldots, \mathbf{w}_N$. In other words,

$$f^{(N)}(\ldots, \mathbf{w}_\alpha, \ldots, \mathbf{w}_\beta, \ldots) = f^{(N)}(\ldots, \mathbf{w}_\beta, \ldots, \mathbf{w}_\alpha, \ldots) \quad \text{for all } \alpha,\ \beta. \tag{7.23}$$

The equation governing the evolution of $f^{(N)}$ is analogous to the collisionless Boltzmann equation governing the evolution of the phase-space density f (§4.1). In fact, to derive the equation for $f^{(N)}$ we need only reinterpret the 3-dimensional vectors $\mathbf{q}$ and $\mathbf{p}$ in that section as $3N$-dimensional vectors $(\mathbf{q}_1, \ldots, \mathbf{q}_N)$, $(\mathbf{p}_1, \ldots, \mathbf{p}_N)$. Then the analogs of equations (4.7)–(4.10) are

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{\alpha=1}^{N} \left(\dot{\mathbf{q}}_\alpha \cdot \frac{\partial f^{(N)}}{\partial \mathbf{q}_\alpha} + \dot{\mathbf{p}}_\alpha \cdot \frac{\partial f^{(N)}}{\partial \mathbf{p}_\alpha} \right) = 0; \tag{7.24}$$

$$\frac{\partial f^{(N)}}{\partial t} + \left[f^{(N)}, H_N \right] = 0; \tag{7.25}$$

$$\frac{\mathrm{d} f^{(N)}}{\mathrm{d} t} = 0; \tag{7.26}$$

where $\mathrm{d}/\mathrm{d}t$ is the convective derivative in Γ-space, and $[\cdot, \cdot]$ denotes the Poisson bracket in Γ-space. In other words the flow of Γ-points through Γ-space is incompressible: the probability density of Γ-points $f^{(N)}$ around the Γ-point of a given system always remains constant. This is **Liouville's theorem**, and equations (7.24)–(7.26) are **Liouville's equation**.[3]

If (i) we work in an inertial frame, (ii) we choose our canonical coordinates and momenta to be the positions $\mathbf{x}_\alpha$ and velocities $\mathbf{v}_\alpha$, and (iii) our particles have mass m and interact only through their mutual gravitation, then Liouville's equation can be written in the form

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{\alpha=1}^{N} \left(\mathbf{v}_\alpha \cdot \frac{\partial f^{(N)}}{\partial \mathbf{x}_\alpha} - \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} \frac{\partial \Phi_{\alpha\beta}}{\partial \mathbf{x}_\alpha} \cdot \frac{\partial f^{(N)}}{\partial \mathbf{v}_\alpha} \right) = 0, \tag{7.27}$$

where $\Phi_{\alpha\beta} = -Gm/|\mathbf{x}_\alpha - \mathbf{x}_\beta|$.

Any N-body DF of the form

$$f^{(N)}(\mathbf{w}_1, \ldots, \mathbf{w}_N) = f[H_N(\mathbf{w}_1, \ldots, \mathbf{w}_N)] \tag{7.28}$$

[3] We adopt the convention that the collisionless Boltzmann equation applies to 6-dimensional phase space and Liouville's equation applies to $6N$-dimensional Γ-space, although some authors use the term Liouville's equation in both cases. With our convention, Liouville's equation is actually not due to Liouville. It was first explicitly derived by Gibbs (1884), two years after Liouville's death. Gibbs was also the first to recognize its importance in astronomy. It might therefore be better called the Gibbs equation.

is a solution of Liouville's equation. The proof is an obvious extension of the Jeans theorem (§4.2). In thermal equilibrium, we would have

$$f^{(N)}(\mathbf{w}_1,\ldots,\mathbf{w}_N) = C\exp[-\beta H_N(\mathbf{w}_1,\ldots,\mathbf{w}_N)], \tag{7.29}$$

where C and β are positive constants. Thermal equilibrium cannot be achieved in a gravitational N-body system because the normalization condition (7.22) cannot be satisfied for a DF of the form (7.29).[4]

7.2.3 Reduced distribution functions

We now investigate how the N-body DF $f^{(N)}(\mathbf{w}_1,\ldots,\mathbf{w}_N,t)$ is related to the usual DF in six-dimensional phase space, $f(\mathbf{w},t)$ (§4.1). We introduce first the **reduced** or **K-body distribution function**, which is obtained by integrating the N-body DF over $N-K$ of the six-vectors $\mathbf{w}_\alpha$. Since $f^{(N)}$ is a symmetric function of the $\mathbf{w}_\alpha$ (eq. 7.23), without loss of generality we may choose the integration variables to be $\mathbf{w}_{K+1},\ldots,\mathbf{w}_N$. Thus we define

$$f^{(K)}(\mathbf{w}_1,\ldots,\mathbf{w}_K,t) \equiv \int \mathrm{d}^6\mathbf{w}_{K+1}\cdots\mathrm{d}^6\mathbf{w}_N\, f^{(N)}(\mathbf{w}_1,\ldots,\mathbf{w}_N,t). \tag{7.30}$$

From equation (7.22), the normalization of the K-body DF is simply

$$\int \mathrm{d}^6\mathbf{w}_1\cdots\mathrm{d}^6\mathbf{w}_K\, f^{(K)}(\mathbf{w}_1,\ldots,\mathbf{w}_K,t) = 1. \tag{7.31}$$

The one-body DF is

$$f^{(1)}(\mathbf{w}_1,t) \equiv \int \mathrm{d}^6\mathbf{w}_2\cdots\mathrm{d}^6\mathbf{w}_N\, f^{(N)}(\mathbf{w}_1,\ldots,\mathbf{w}_N,t). \tag{7.32}$$

The one-body DF describes the probability of finding a particular star in a unit volume of phase space centered on $\mathbf{w}_1$. This is the same as the definition of the phase-space DF in §4.1, and therefore we are free to simplify our notation by writing

$$f(\mathbf{w},t) = f^{(1)}(\mathbf{w},t). \tag{7.33}$$

In many situations, it is useful to write the two-body DF in the form

$$f^{(2)}(\mathbf{w}_1,\mathbf{w}_2,t) = f(\mathbf{w}_1,t)f(\mathbf{w}_2,t) + g(\mathbf{w}_1,\mathbf{w}_2,t). \tag{7.34}$$

[4] The integral diverges at both large and small scales. When the particles are separated by large distances, $f^{(N)}$ depends on velocity but is independent of position, so the spatial integral diverges. When two particles α and β approach one another, $\Phi_{\alpha\beta}$ diverges so $\exp(-\beta H_N)$ becomes extremely large.

The function g is called the **two-body correlation function**; the terminology is borrowed from probability theory, where the variables x and y are said to be uncorrelated if the joint probability $p(x, y)$ can be factored into a product of the form $p_x(x)p_y(y)$. Loosely speaking, the two-body correlation function measures the excess probability of finding a particle at $\mathbf{w}_1$ due to the presence of a particle at $\mathbf{w}_2$. A more precise statement can be derived from the laws of conditional probability (eq. B.85), which state that the probability that a star is located in a unit volume of phase space centered on $\mathbf{w}_1$, given that a star is known to be located at $\mathbf{w}_2$, is

$$f(\mathbf{w}_1|\mathbf{w}_2) = \frac{f^{(2)}(\mathbf{w}_1, \mathbf{w}_2)}{\int \mathrm{d}^6\mathbf{w}_1' f^{(2)}(\mathbf{w}_1', \mathbf{w}_2)} = \frac{f(\mathbf{w}_1)f(\mathbf{w}_2) + g(\mathbf{w}_1, \mathbf{w}_2)}{f(\mathbf{w}_2) + \int \mathrm{d}^6\mathbf{w}_1'\, g(\mathbf{w}_1', \mathbf{w}_2)}. \qquad (7.35)$$

In particular, if the correlation function $g(\mathbf{w}_1, \mathbf{w}_2) = 0$, then $f(\mathbf{w}_1|\mathbf{w}_2) = f(\mathbf{w}_1)$; in other words, the presence of a star at $\mathbf{w}_2$ has no effect on the probability of finding a star near $\mathbf{w}_1$.

The use of reduced DFs can be illustrated by computing the expectation value of the kinetic and potential energy for a stellar system. From equation (7.21), the expectation of the kinetic energy is

$$\langle K \rangle = \tfrac{1}{2}m \int \mathrm{d}^6\mathbf{w}_1 \cdots \mathrm{d}^6\mathbf{w}_N\, f^{(N)}(\mathbf{w}_1, \ldots, \mathbf{w}_N, t) \sum_{\alpha=1}^{N} v_\alpha^2; \qquad (7.36)$$

since the stars are identical, this simplifies to

$$\langle K \rangle = \tfrac{1}{2}Nm \int \mathrm{d}^6\mathbf{w}_1\, f(\mathbf{w}_1, t) v_1^2. \qquad (7.37)$$

Similarly, *any* observable that involves only quantities that depend additively on the phase-space coordinates of single stars can be expressed in terms of the one-body DF. Such observables include density, surface brightness, line-of-sight velocity distribution, metallicity distribution, etc.

The expectation of the potential energy is

$$\langle W \rangle = -\tfrac{1}{2} \int \mathrm{d}^6\mathbf{w}_1 \cdots \mathrm{d}^6\mathbf{w}_N\, f^{(N)}(\mathbf{w}_1, \ldots, \mathbf{w}_N, t) \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{Gm^2}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|}. \qquad (7.38)$$

Since the stars are identical, and there are $N(N-1)$ ways in which we can choose two distinct stars α and β from N, this simplifies to

$$\langle W \rangle = -\tfrac{1}{2}Gm^2 N(N-1) \int \mathrm{d}^6\mathbf{w}_1 \mathrm{d}^6\mathbf{w}_2 \frac{f^{(2)}(\mathbf{w}_1, \mathbf{w}_2, t)}{|\mathbf{x}_2 - \mathbf{x}_1|}. \qquad (7.39)$$

Thus the potential energy depends only on the two-body DF. If the correlation function is small, that is, if $|g(\mathbf{w}_1, \mathbf{w}_2, t)| \ll f(\mathbf{w}_1)f(\mathbf{w}_2)$, then for $N \gg 1$ the potential energy simplifies to

$$W = -\tfrac{1}{2}Gm^2N^2 \int \mathrm{d}^6\mathbf{w}_1\mathrm{d}^6\mathbf{w}_2 \frac{f(\mathbf{w}_1,t)f(\mathbf{w}_2,t)}{|\mathbf{x}_2 - \mathbf{x}_1|} = \tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})\Phi(\mathbf{x}), \tag{7.40}$$

which is the expression we have used in prior chapters (eq. 2.18).

7.2.4 Relation of Liouville's equation to the collisionless Boltzmann equation

The N-body DF is said to be **separable** if it is simply the product of one-body DFs, that is, if

$$f^{(N)}(\mathbf{w}_1, \dots, \mathbf{w}_N, t) = \prod_{\beta=1}^{N} f(\mathbf{w}_\beta, t). \tag{7.41}$$

As we have seen, this assumption implies that the positions of stars are uncorrelated, in the sense that the probability of finding a star near any phase-space position $\mathbf{w}_1$ is unaffected by the presence or absence of stars at nearby points. We now assume that the N-body DF is separable, and ask for the equation governing the evolution of the one-body DF f. To find this, we integrate Liouville's equation (7.27) over $\mathrm{d}^6\mathbf{w}_2 \cdots \mathrm{d}^6\mathbf{w}_N$. The term involving $\partial f^{(N)}/\partial t$ simply yields $\partial f(\mathbf{w}_1,t)/\partial t$. The term involving $\partial f^{(N)}/\partial \mathbf{x}_\alpha$ yields zero if $\alpha = 2, \dots, N$ because $\int \mathrm{d}^3\mathbf{x}_\alpha\, \partial f^{(N)}/\partial \mathbf{x}_\alpha = 0$ so long as $f^{(N)} \to 0$ sufficiently fast as $|\mathbf{x}_\alpha| \to \infty$. The integration of the term involving $\partial f^{(N)}/\partial \mathbf{v}_\alpha$ yields zero if $\alpha = 2, \dots, N$ for a similar reason. Thus we obtain

$$\begin{aligned} &\frac{\partial f(\mathbf{w}_1,t)}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f(\mathbf{w}_1,t)}{\partial \mathbf{x}_1} \\ &\qquad - \frac{\partial f(\mathbf{w}_1,t)}{\partial \mathbf{v}_1} \cdot \sum_{\beta=2}^{N} \int \mathrm{d}^6\mathbf{w}_2 \cdots \mathrm{d}^6\mathbf{w}_N \frac{\partial \Phi_{1\beta}}{\partial \mathbf{x}_1} \prod_{\alpha=2}^{N} f(\mathbf{w}_\alpha, t) = 0. \end{aligned} \tag{7.42}$$

Each term in the sum is identical, and $\int \mathrm{d}^6\mathbf{w}\, f(\mathbf{w},t) = 1$, so this becomes

$$\frac{\partial f(\mathbf{w}_1,t)}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f(\mathbf{w}_1,t)}{\partial \mathbf{x}_1} - (N-1)\frac{\partial f(\mathbf{w}_1,t)}{\partial \mathbf{v}_1} \cdot \int \mathrm{d}^6\mathbf{w}_2 \frac{\partial \Phi_{12}}{\partial \mathbf{x}_1} f(\mathbf{w}_2,t) = 0. \tag{7.43}$$

The expectation value of the gravitational potential at $\mathbf{x}_1$ is

$$\overline{\Phi}(\mathbf{x}_1, t) = N \int \mathrm{d}^6\mathbf{w}_2\, \Phi_{12} f(\mathbf{w}_2, t), \tag{7.44}$$

so equation (7.43) simplifies to

$$\frac{\partial f(\mathbf{w},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f(\mathbf{w},t)}{\partial \mathbf{x}} - \frac{N-1}{N}\frac{\partial\overline{\Phi}(\mathbf{x},t)}{\partial \mathbf{x}}\frac{\partial f(\mathbf{w},t)}{\partial \mathbf{v}} = 0. \tag{7.45}$$

In the limit $N \to \infty$ this becomes the collisionless Boltzmann equation (4.11). Thus we have shown that *the collisionless Boltzmann equation results from the Liouville equation when $N \gg 1$ and the N-body* DF *is separable.*

If the DF is *not* separable and $N \gg 1$, it is straightforward to show that equation (7.45) must be replaced by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \Gamma[f], \tag{7.46}$$

where $\Gamma[f]$ is the **encounter operator**, given by

$$\Gamma[f(\mathbf{w}_1,t)] \equiv N\int \mathrm{d}^6\mathbf{w}_2 \frac{\partial\Phi_{12}}{\partial\mathbf{x}_1}\cdot\frac{\partial g(\mathbf{w}_1,\mathbf{w}_2,t)}{\partial\mathbf{v}_1}, \tag{7.47}$$

and g is the two-body correlation function (eq. 7.34). Thus the correlations between particles in phase space, as measured by $g(\mathbf{w}_1,\mathbf{w}_2,t)$, drive the rate of change of the phase-space density around a given star, given by $\Gamma[f]$.

We can determine the encounter operator $\Gamma[f]$ in two ways. The first approach is through the correlation function. Just as we derived equation (7.46) for the one-body DF by integrating Liouville's equation over $\mathrm{d}^6\mathbf{w}_2\cdots\mathrm{d}^6\mathbf{w}_N$, we can derive an equation for the correlation function—or, what is equivalent, the two-body DF $f^{(2)}(\mathbf{w}_1,\mathbf{w}_2,t)$—by integrating Liouville's equation over $\mathrm{d}^6\mathbf{w}_3\cdots\mathrm{d}^6\mathbf{w}_N$. Unfortunately, just as equation (7.46) for the one-body DF depends on the two-body DF through the appearance of $g(\mathbf{w}_1,\mathbf{w}_2,t)$ on the right side, the equation for the two-body DF depends on the three-body DF.[5] However, in the limit where the number of stars $N \to \infty$, while the total mass mN remains constant, we can neglect the contribution of the three-body correlation function to the equation governing the two-body DF. The resulting equation can be solved to determine the two-body correlation function, and this can be substituted into equation (7.47) to determine the encounter operator (Gilbert 1968; Lifshitz & Pitaevskii 1981).

A more physical approach, which we take in §7.4, is to ask how encounters between stars modify the one-body DF. This approach is only practical when the encounters can be approximated as localized in both time and space; fortunately, we shall see that this approximation is remarkably accurate for most stellar systems.

[5] Continuing in this way, we would obtain a sequence of equations of rapidly increasing complexity, each expressing the rate of change of $f^{(n)}$ in terms of $f^{(n+1)}$. This sequence is known as the **BBGKY hierarchy**, after N. N. Bogoliubov, M. Born and H. S. Green, J. G. Kirkwood, and J. Yvon, who all discovered the equations independently between 1935 and 1946.