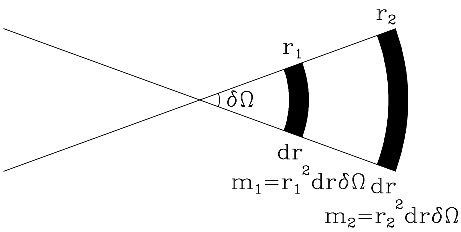

Figure 1.4 If the density of stars were everywhere the same, the stars in each of the shaded segments of a cone would contribute equally to the force on a star at the cone's apex. Thus the acceleration of a star at the apex is determined mainly by the large-scale distribution of stars in the galaxy, not by the star's nearest neighbors.

the star at the apex with a force proportional to $r^{-2} \times r^2 \times r = r$. This simple argument shows that the force on the star at the apex is dominated by the most distant stars in the system, rather than by its closest neighbors. Of course, if the density of attracting stars were exactly spherical, the star at the apex would experience no net force because it would be pulled equally in all directions. But in general the density of attracting stars falls off in one direction more slowly than in the opposing direction, so the star at the apex is subject to a net force, and this force is determined by the structure of the galaxy on the largest scale. Consequently—in contrast to the situation for molecules—the force on a star does not vary rapidly, and each star may be supposed to accelerate smoothly through the force field that is generated by the galaxy as a whole. In other words, for most purposes we can treat the gravitational force on a star as arising from a smooth density distribution rather than a collection of mass points.

1.2.1 The relaxation time

We now investigate this conclusion more quantitatively, by asking how accurately we can approximate a galaxy composed of N identical stars of mass m as a smooth density distribution and gravitational field. To answer this question, we follow the motion of an individual star, called the **subject star**, as its orbit carries it once across the galaxy, and seek an order-of-magnitude estimate of the difference between the actual velocity of this star after this interval and the velocity that it would have had if the mass of the other stars were smoothly distributed. Suppose the subject star passes within distance b of another star, called the **field star** (Figure 1.5). We want to estimate the amount $\delta\mathbf{v}$ by which the encounter deflects the velocity $\mathbf{v}$ of the subject star. In §3.1d we calculate $\delta\mathbf{v}$ exactly, but for our present purposes an approximate estimate is sufficient. To make this estimate we shall assume that $|\delta\mathbf{v}|/v \ll 1$, and that the field star is stationary during the encounter. In this case $\delta\mathbf{v}$ is perpendicular to $\mathbf{v}$, since the accelerations parallel to $\mathbf{v}$ average to zero. We may calculate the magnitude of the velocity change, $\delta v \equiv |\delta\mathbf{v}|$, by assuming that the subject star passes the field star on a straight-line trajectory, and integrating the perpendicular force $F_\perp$ along this trajectory. We place the origin of time at the instant of closest approach of the two stars,

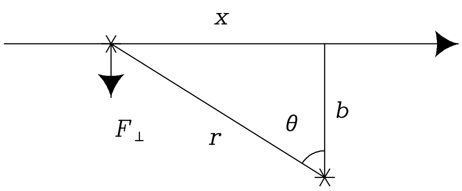

Figure 1.5 A field star approaches the subject star at speed v and impact parameter b. We estimate the resulting impulse to the subject star by approximating the field star's trajectory as a straight line.

and find in the notation of Figure 1.5,

$$F_\perp = \frac{Gm^2}{b^2+x^2}\cos\theta = \frac{Gm^2 b}{(b^2+x^2)^{3/2}} = \frac{Gm^2}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}. \tag{1.28}$$

But by Newton's laws

$$m\dot{\mathbf{v}} = \mathbf{F} \quad \text{so} \quad \delta v = \frac{1}{m}\int_{-\infty}^{\infty} \mathrm{d}t\, F_\perp, \tag{1.29}$$

and we have

$$\delta v = \frac{Gm}{b^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{[1+(vt/b)^2]^{3/2}} = \frac{Gm}{bv}\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(1+s^2)^{3/2}} = \frac{2Gm}{bv}. \tag{1.30}$$

In words, δv is roughly equal to the acceleration at closest approach, Gm/b^2, times the duration of this acceleration $2b/v$. Notice that our assumption of a straight-line trajectory breaks down, and equation (1.30) becomes invalid, when $\delta v \simeq v$; from equation (1.30), this occurs if the impact parameter $b \lesssim b_{90} \equiv 2Gm/v^2$. The subscript 90 stands for a 90-degree deflection—see equation (3.51) for a more precise definition.

Now the surface density of field stars in the host galaxy is of order $N/\pi R^2$, where N is the number of stars and R is the galaxy's radius, so in crossing the galaxy once the subject star suffers

$$\delta n = \frac{N}{\pi R^2} 2\pi b\,\mathrm{d}b = \frac{2N}{R^2} b\,\mathrm{d}b \tag{1.31}$$

encounters with impact parameters in the range b to $b+\mathrm{d}b$. Each such encounter produces a perturbation $\delta\mathbf{v}$ to the subject star's velocity, but because these small perturbations are randomly oriented in the plane perpendicular to $\mathbf{v}$, their mean is zero.[10] Although the mean velocity change is zero, the mean-square change is not: after one crossing this amounts to

$$\sum \delta v^2 \simeq \delta v^2 \delta n = \left(\frac{2Gm}{bv}\right)^2 \frac{2N}{R^2} b\,\mathrm{d}b. \tag{1.32}$$

[10] Strictly, the mean change in velocity is zero only if the distribution of perturbing stars is the same in all directions. A more precise statement is that the mean change in velocity is due to the smoothed-out mass distribution, and we ignore this because the goal of our calculation is to determine the *difference* between the acceleration due to the smoothed mass distribution and the actual stars.

Integrating equation (1.32) over all impact parameters from $b_{\rm min}$ to $b_{\rm max}$, we find the mean-square velocity change per crossing,

$$\Delta v^2 \equiv \int_{b_{\rm min}}^{b_{\rm max}} \sum \delta v^2 \simeq 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda, \tag{1.33a}$$

where the factor

$$\ln\Lambda \equiv \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \tag{1.33b}$$

is called the **Coulomb logarithm**. Our assumption of a straight-line trajectory breaks down for impact parameters smaller than b_{90}, so we set $b_{\rm min} = f_1 b_{90}$, where f_1 is a factor of order unity. Our assumption of a homogeneous distribution of field stars breaks down for impact parameters of order R, so we set $b_{\rm max} = f_2 R$. Then

$$\ln\Lambda = \ln\left(\frac{R}{b_{90}}\right) + \ln(f_2/f_1). \tag{1.34}$$

In most systems of interest $R \gg b_{90}$ (for example, in a typical elliptical galaxy $R/b_{90} \gtrsim 10^{10}$), so the fractional uncertainty in $\ln\Lambda$ arising from the uncertain values of f_1 and f_2 is quite small, and we lose little accuracy by setting $f_2/f_1 = 1$.

Thus encounters between the subject star and field stars cause a kind of diffusion of the subject star's velocity that is distinct from the steady acceleration caused by the overall mass distribution in the stellar system. This diffusive process is sometimes called **two-body relaxation** since it arises from the cumulative effect of myriad two-body encounters between the subject star and passing field stars.

The typical speed v of a field star is roughly that of a particle in a circular orbit at the edge of the galaxy,

$$v^2 \approx \frac{GNm}{R}. \tag{1.35}$$

If we eliminate R from equation (1.33a) using equation (1.35), we have

$$\frac{\Delta v^2}{v^2} \approx \frac{8\ln\Lambda}{N}. \tag{1.36}$$

If the subject star makes many crossings of the galaxy, the velocity $\mathbf{v}$ will change by roughly Δv^2 at each crossing, so the number of crossings $n_{\rm relax}$ that is required for its velocity to change by of order itself is given by

$$n_{\rm relax} \simeq \frac{N}{8\ln\Lambda}. \tag{1.37}$$

The **relaxation time** may be defined as $t_{\rm relax} = n_{\rm relax} t_{\rm cross}$, where $t_{\rm cross} = R/v$ is the **crossing time**, the time needed for a typical star to cross the galaxy once. Moreover $\Lambda = R/b_{90} \approx Rv^2/(Gm)$, which is $\approx N$ by equation (1.35). Thus our final result is

$$t_{\rm relax} \simeq \frac{0.1N}{\ln N} t_{\rm cross}. \tag{1.38}$$

After one relaxation time, the cumulative small kicks from many encounters with passing stars have changed the subject star's orbit significantly from the one it would have had if the gravitational field had been smooth. In effect, after a relaxation time a star has lost its memory of its initial conditions. Galaxies typically have $N \approx 10^{11}$ stars and are a few hundred crossing times old, so for these systems stellar encounters are unimportant, except very near their centers. In a globular cluster, on the other hand, $N \approx 10^5$ and the crossing time $t_{\rm cross} \approx 1\,{\rm Myr}$ (Table 1.3), so relaxation strongly influences the cluster structure over its lifetime of 10 Gyr.

In all of these systems the dynamics over timescales $\lesssim t_{\rm relax}$ is that of a **collisionless system** in which the constituent particles move under the influence of the gravitational field generated by a smooth mass distribution, rather than a collection of mass points. Non-baryonic dark matter is also collisionless, since both weak interactions and gravitational interactions between individual WIMPs are negligible in any galactic context.

In most of this book we focus on collisionless stellar dynamics, confining discussion of the longer-term evolution that is driven by gravitational encounters among the particles to Chapter 7.

1.3 The cosmological context

This section provides a summary of the aspects of cosmology that we use in this book. For more information the reader can consult texts such as Weinberg (1972), Peebles (1993), and Peacock (1999).

To a very good approximation, the universe is observed to be homogeneous and isotropic on large scales—here "large" means $\gtrsim 100\,{\rm Mpc}$, which is still much smaller than the characteristic "size" of the universe, the **Hubble length** $c/H_0 = 4.3h_7^{-1}\,{\rm Gpc}$ where $1\,{\rm Gpc} = 10^9\,{\rm pc} = 10^3\,{\rm Mpc}$ and c is the speed of light. Therefore a useful first approximation is to average over the small-scale structure and treat the universe as *exactly* homogeneous and isotropic. Of course, the universe does not appear isotropic to all observers: an observer traveling rapidly with respect to the local matter will see galaxies approaching in one direction and receding in another. Therefore we define a set of **fundamental observers**, who are at rest with respect to the matter around them.[11] The universe is expanding, so we may synchronize the

[11] A more precise definition is that a fundamental observer sees no dipole component in the cosmic microwave background radiation (§1.3.5).