The discussion in §§2.4 to 2.6 is rather mathematical, and readers who are willing to take a few results on trust may prefer to move straight from §2.3 to §2.7.

## 2.1 General results

Our goal is to calculate the force $\mathbf{F}(\mathbf{x})$ on a particle of mass $m_{\rm s}$ at position $\mathbf{x}$ that is generated by the gravitational attraction of a distribution of mass $\rho(\mathbf{x}')$. According to Newton's inverse-square law of gravitation, the force $\mathbf{F}(\mathbf{x})$ may be obtained by summing the small contributions

$$\delta\mathbf{F}(\mathbf{x}) = Gm_{\rm s}\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\delta m(\mathbf{x}') = Gm_{\rm s}\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}')\,{\rm d}^3\mathbf{x}' \tag{2.1}$$

to the overall force from each small element of volume ${\rm d}^3\mathbf{x}'$ located at $\mathbf{x}'$. Thus

$$\mathbf{F}(\mathbf{x}) = m_{\rm s}\mathbf{g}(\mathbf{x}) \quad \text{where} \quad \mathbf{g}(\mathbf{x}) \equiv G\int {\rm d}^3\mathbf{x}'\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}') \tag{2.2}$$

is the **gravitational field**, the force per unit mass.

If we define the **gravitational potential** $\Phi(\mathbf{x})$ by

$$\Phi(\mathbf{x}) \equiv -G\int {\rm d}^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}, \tag{2.3}$$

and notice that

$$\boldsymbol{\nabla}_\mathbf{x}\left(\frac{1}{|\mathbf{x}'-\mathbf{x}|}\right) = \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}, \tag{2.4}$$

we find that we may write $\mathbf{g}$ as

$$\begin{aligned}\mathbf{g}(\mathbf{x}) &= \boldsymbol{\nabla}_\mathbf{x}\int {\rm d}^3\mathbf{x}'\frac{G\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|} \\ &= -\boldsymbol{\nabla}\Phi,\end{aligned} \tag{2.5}$$

where for brevity we have dropped the subscript $\mathbf{x}$ on the gradient operator $\boldsymbol{\nabla}$.

The potential is useful because it is a scalar field that is easier to visualize than the vector gravitational field but contains the same information. Also, in many situations the easiest way to obtain $\mathbf{g}$ is first to calculate the potential and then to take its gradient.

If we take the divergence of equation (2.2), we find

$$\boldsymbol{\nabla}\cdot\mathbf{g}(\mathbf{x}) = G\int {\rm d}^3\mathbf{x}'\,\boldsymbol{\nabla}_\mathbf{x}\cdot\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\right)\rho(\mathbf{x}'). \tag{2.6}$$

Now

$$\nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = -\frac{3}{|\mathbf{x}' - \mathbf{x}|^3} + \frac{3(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^5}. \tag{2.7}$$

When $\mathbf{x}' - \mathbf{x} \neq 0$ we may cancel the factor $|\mathbf{x}' - \mathbf{x}|^2$ from top and bottom of the last term in this equation to conclude that

$$\nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = 0 \qquad (\mathbf{x}' \neq \mathbf{x}). \tag{2.8}$$

Therefore, any contribution to the integral of equation (2.6) must come from the point $\mathbf{x}' = \mathbf{x}$, and we may restrict the volume of integration to a small sphere of radius $h$ centered on this point. Since, for sufficiently small $h$, the density will be almost constant through this volume, we can take $\rho(\mathbf{x}')$ out of the integral. The remaining terms of the integrand may then be arranged as follows:

$$\begin{aligned} \nabla \cdot \mathbf{g}(\mathbf{x}) &= G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| \le h} \mathrm{d}^3\mathbf{x}' \, \nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \\ &= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| \le h} \mathrm{d}^3\mathbf{x}' \, \nabla_{\mathbf{x}'} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \\ &= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| = h} \mathrm{d}^2\mathbf{S}' \cdot \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3}. \end{aligned} \tag{2.9a}$$

The last step in this sequence uses the divergence theorem to convert the volume integral into a surface integral (eq. B.43). Now on the sphere $|\mathbf{x}' - \mathbf{x}| = h$ we have $\mathrm{d}^2\mathbf{S}' = (\mathbf{x}' - \mathbf{x})h \, \mathrm{d}^2\Omega$, where $\mathrm{d}^2\Omega$ is a small element of solid angle. Hence equation (2.9a) becomes

$$\nabla \cdot \mathbf{g}(\mathbf{x}) = -G\rho(\mathbf{x}) \int \mathrm{d}^2\Omega = -4\pi G\rho(\mathbf{x}). \tag{2.9b}$$

If we substitute from equation (2.5) for $\nabla \cdot \mathbf{g}$, we obtain **Poisson's equation** relating the potential $\Phi$ to the density $\rho$;

$$\nabla^2\Phi = 4\pi G\rho. \tag{2.10}$$

This is a differential equation that can be solved for $\Phi(\mathbf{x})$ given $\rho(\mathbf{x})$ and an appropriate boundary condition.[1] For an isolated system the boundary condition is $\Phi \to 0$ as $|\mathbf{x}| \to \infty$. The potential given by equation (2.3) automatically satisfies this boundary condition. Poisson's equation provides

[1] Using the physically correct boundary condition is essential: for example, if $\Phi(\mathbf{x})$ is a solution, then so is $\Phi(\mathbf{x}) + \mathbf{k} \cdot \mathbf{x}$, with $\mathbf{k}$ an arbitrary constant vector, but the corresponding gravitational fields differ by $\mathbf{k}$.

a route to $\Phi$ and then to $\mathbf{g}$ that is often more convenient than equations (2.2) or (2.3). In the special case $\rho = 0$ Poisson's equation becomes **Laplace's equation**,

$$\nabla^2\Phi = 0. \tag{2.11}$$

If we integrate both sides of equation (2.10) over an arbitrary volume containing total mass $M$, and then apply the divergence theorem (eq. B.43), we obtain

$$4\pi GM = 4\pi G\int \mathrm{d}^3\mathbf{x}\,\rho = \int \mathrm{d}^3\mathbf{x}\,\nabla^2\Phi = \int \mathrm{d}^2\mathbf{S}\cdot\boldsymbol{\nabla}\Phi. \tag{2.12}$$

This result is **Gauss's theorem**, which states that *the integral of the normal component of $\boldsymbol{\nabla}\Phi$ over any closed surface equals $4\pi G$ times the mass contained within that surface.*

Since $\mathbf{g}$ is determined by the gradient of a potential, the gravitational field is conservative, that is, the work done against gravitational forces in moving two stars from infinity to a given configuration is independent of the path along which they are moved, and is defined to be the potential energy of the configuration (Appendix D.1). Similarly, the work done against gravitational forces in assembling an arbitrary continuous distribution of mass $\rho(\mathbf{x})$ is independent of the details of how the mass distribution was assembled, and is defined to be equal to the **potential energy** of the mass distribution. An expression for the potential energy can be obtained by the following argument.

Suppose that some of the mass is already in place so that the density and potential are $\rho(\mathbf{x})$ and $\Phi(\mathbf{x})$. If we now bring in a additional small mass $\delta m$ from infinity to position $\mathbf{x}$, the work done is $\delta m\,\Phi(\mathbf{x})$. Thus, if we add a small increment of density $\delta\rho(\mathbf{x})$, the change in potential energy is

$$\delta W = \int \mathrm{d}^3\mathbf{x}\,\delta\rho(\mathbf{x})\Phi(\mathbf{x}). \tag{2.13}$$

According to Poisson's equation the resulting change in potential $\delta\Phi(\mathbf{x})$ satisfies $\nabla^2(\delta\Phi) = 4\pi G(\delta\rho)$, so

$$\delta W = \frac{1}{4\pi G}\int \mathrm{d}^3\mathbf{x}\,\Phi\nabla^2(\delta\Phi)\,. \tag{2.14}$$

Using the divergence theorem in the form (B.45), we may write this as

$$\delta W = \frac{1}{4\pi G}\int \Phi\boldsymbol{\nabla}(\delta\Phi)\cdot\mathrm{d}^2\mathbf{S} - \frac{1}{4\pi G}\int \mathrm{d}^3\mathbf{x}\,\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}(\delta\Phi), \tag{2.15}$$

where the surface integral vanishes because $\Phi \propto r^{-1}$, $|\boldsymbol{\nabla}\delta\Phi| \propto r^{-2}$ as $r\to\infty$, so the integrand $\propto r^{-3}$ while the total surface area $\propto r^2$. But $\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}(\delta\Phi) = \frac{1}{2}\delta(\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi) = \frac{1}{2}\delta|(\boldsymbol{\nabla}\Phi)|^2$. Hence

$$\delta W = -\frac{1}{8\pi G}\delta\left(\int \mathrm{d}^3\mathbf{x}\,|\boldsymbol{\nabla}\Phi|^2\right). \tag{2.16}$$

If we now sum up all of the contributions $\delta W$, we have a simple expression for the potential energy,

$$W = -\frac{1}{8\pi G}\int \mathrm{d}^3\mathbf{x}\,|\boldsymbol{\nabla}\Phi|^2. \tag{2.17}$$

To obtain an alternative expression for $W$, we again apply the divergence theorem and replace $\nabla^2\Phi$ by $4\pi G\rho$ to obtain

$$W = \tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})\Phi(\mathbf{x}). \tag{2.18}$$

**The potential-energy tensor** In §4.8.3 we shall encounter the tensor $\mathbf{W}$ that is defined by

$$W_{jk} \equiv -\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})x_j\frac{\partial\Phi}{\partial x_k}, \tag{2.19}$$

where $\rho$ and $\Phi$ are the density and potential of some body, and the integral is to be taken over all space. We now deduce some useful properties of $\mathbf{W}$, which is known as the **Chandrasekhar potential-energy tensor**.[2]

If we substitute for $\Phi$ from equation (2.3), $\mathbf{W}$ becomes

$$W_{jk} = G\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})x_j\frac{\partial}{\partial x_k}\int \mathrm{d}^3\mathbf{x}'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}. \tag{2.20}$$

Since the range of the integration over $\mathbf{x}'$ does not depend on $\mathbf{x}$, we may carry the differentiation inside the integral to find

$$W_{jk} = G\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}'\,\rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_j(x'_k-x_k)}{|\mathbf{x}'-\mathbf{x}|^3}. \tag{2.21a}$$

Furthermore, since $\mathbf{x}$ and $\mathbf{x}'$ are dummy variables of integration, we may relabel them and write

$$W_{jk} = G\int \mathrm{d}^3\mathbf{x}'\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x}')\rho(\mathbf{x})\frac{x'_j(x_k-x'_k)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{2.21b}$$

Finally, on interchanging the order of integration in equation (2.21b) and adding the result to equation (2.21a), we obtain

$$W_{jk} = -\tfrac{1}{2}G\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}'\,\rho(\mathbf{x})\rho(\mathbf{x}')\frac{(x'_j-x_j)(x'_k-x_k)}{|\mathbf{x}'-\mathbf{x}|^3}. \tag{2.22}$$

[2] Subramanyan Chandrasekhar (1910–1995) was educated in India and England and spent most of his career at the University of Chicago. He discovered the Chandrasekhar limit, the maximum mass of a white dwarf star, and elucidated the concept of dynamical friction in astrophysics (§8.1). He shared the 1983 Nobel Prize in Physics with W. A. Fowler.

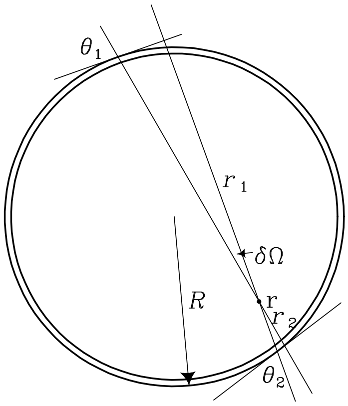


**Figure 2.1** Proof of Newton's first theorem.

From this expression we draw the important inference that the tensor $\mathbf{W}$ is **symmetric**, that is, that $W_{jk} = W_{kj}$. If the body is flattened along some axis, say the $x_3$ axis, $W_{33}$ will be smaller than the other components because for most pairs of matter elements, $|x_3 - x_3'| < |x_1 - x_1'|$ or $|x_2 - x_2'|$.

When we take the **trace** of both sides of equation (2.22), we find

$$\begin{aligned}\mathrm{trace}(\mathbf{W}) \equiv \sum_{j=1}^{3} W_{jj} &= -\tfrac{1}{2}G \int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x}) \int \mathrm{d}^3\mathbf{x}'\, \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \\ &= \tfrac{1}{2} \int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x})\Phi(\mathbf{x}).\end{aligned} \tag{2.23}$$

Comparing this with equation (2.18) we see that trace($\mathbf{W}$) is simply the total gravitational potential energy $W$. Taking the trace of (2.19) we have

$$W = -\int \mathrm{d}^3\mathbf{x}\,\rho\mathbf{x}\cdot\boldsymbol{\nabla}\Phi, \tag{2.24}$$

which provides another useful expression for the potential energy of a body.

## 2.2 Spherical systems

### 2.2.1 Newton's theorems

Newton proved two results that enable us to calculate the gravitational potential of any spherically symmetric distribution of matter easily:

**Newton's first theorem** *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*

**Newton's second theorem** *The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.*