But from either (3.47) or (3.48) we see that the point of closest approach is reached when $\psi = \psi_0$. Since the orbit is symmetrical about this point, the angle through which the reduced particle's velocity is deflected is $\theta_{\mathrm{defl}} = 2\psi_0 - \pi$ (see Figure 3.2). It proves useful to define the **90° deflection radius** as the impact parameter at which $\theta_{\mathrm{defl}} = 90°$:

$$b_{90} \equiv \frac{G(M+m)}{V_0^2}. \tag{3.51}$$

Thus

$$\theta_{\mathrm{defl}} = 2\tan^{-1}\left(\frac{G(M+m)}{bV_0^2}\right) = 2\tan^{-1}(b_{90}/b). \tag{3.52}$$

By conservation of energy, the relative speed after the encounter equals the initial speed $V_0$. Hence the components $\Delta\mathbf{V}_\parallel$ and $\Delta\mathbf{V}_\perp$ of $\Delta\mathbf{V}$ parallel and perpendicular to the original relative velocity vector $\mathbf{V}_0$ are given by

$$\begin{aligned} |\Delta\mathbf{V}_\perp| &= V_0 \sin\theta_{\mathrm{defl}} = V_0|\sin 2\psi_0| = \frac{2V_0|\tan\psi_0|}{1+\tan^2\psi_0} \\ &= \frac{2V_0(b/b_{90})}{1+b^2/b_{90}^2}, \end{aligned} \tag{3.53a}$$

$$\begin{aligned} |\Delta\mathbf{V}_\parallel| &= V_0(1-\cos\theta_{\mathrm{defl}}) = V_0(1+\cos 2\psi_0) = \frac{2V_0}{1+\tan^2\psi_0} \\ &= \frac{2V_0}{1+b^2/b_{90}^2}. \end{aligned} \tag{3.53b}$$

$\Delta\mathbf{V}_\parallel$ always points in the direction opposite to $\mathbf{V}_0$. By equation (3.45) we obtain the components of $\Delta\mathbf{v}_M$ as

$$|\Delta\mathbf{v}_{M\perp}| = \frac{2mV_0}{M+m}\frac{b/b_{90}}{1+b^2/b_{90}^2}, \tag{3.54a}$$

$$|\Delta\mathbf{v}_{M\parallel}| = \frac{2mV_0}{M+m}\frac{1}{1+b^2/b_{90}^2}. \tag{3.54b}$$

$\Delta\mathbf{v}_{M\parallel}$ always points in the direction opposite to $\mathbf{V}_0$. Notice that in the limit of large impact parameter $b$, $|\Delta\mathbf{v}_{M\perp}| = 2Gm/(bV_0)$, which agrees with the determination of the same quantity in equation (1.30).

### 3.1.1 Constants and integrals of the motion

Any stellar orbit traces a path in the six-dimensional space for which the coordinates are the position and velocity $\mathbf{x}, \mathbf{v}$. This space is called **phase space**.[1] A **constant of motion** in a given force field is any function

[1] In statistical mechanics phase space usually refers to position-momentum space rather than position-velocity space. Since all bodies have the same acceleration in a given gravitational field, mass is irrelevant, and position-velocity space is more convenient.

$C(\mathbf{x}, \mathbf{v}; t)$ of the phase-space coordinates and time that is constant along stellar orbits; that is, if the position and velocity along an orbit are given by $\mathbf{x}(t)$ and $\mathbf{v}(t) = \mathrm{d}\mathbf{x}/\mathrm{d}t$,

$$C[\mathbf{x}(t_1), \mathbf{v}(t_1); t_1] = C[\mathbf{x}(t_2), \mathbf{v}(t_2); t_2] \tag{3.55}$$

for any $t_1$ and $t_2$.

An **integral of motion** $I(\mathbf{x}, \mathbf{v})$ is any function of the phase-space coordinates alone that is constant along an orbit:

$$I[\mathbf{x}(t_1), \mathbf{v}(t_1)] = I[\mathbf{x}(t_2), \mathbf{v}(t_2)]. \tag{3.56}$$

While every integral is a constant of the motion, the converse is not true. For example, on a circular orbit in a spherical potential the azimuthal coordinate $\psi$ satisfies $\psi = \Omega t + \psi_0$, where $\Omega$ is the star's constant angular speed and $\psi_0$ is its azimuth at $t = 0$. Hence $C(\psi, t) \equiv t - \psi/\Omega$ is a constant of the motion, but it is not an integral because it depends on time as well as the phase-space coordinates.

Any orbit in any force field always has six independent constants of motion. Indeed, since the initial phase-space coordinates $(\mathbf{x}_0, \mathbf{v}_0) \equiv [\mathbf{x}(0), \mathbf{v}(0)]$ can always be determined from $[\mathbf{x}(t), \mathbf{v}(t)]$ by integrating the equations of motion backward, $(\mathbf{x}_0, \mathbf{v}_0)$ can be regarded as six constants of motion.

By contrast, orbits can have from zero to five integrals of motion. In certain important cases, a few of these integrals can be written down easily: in any static potential $\Phi(\mathbf{x})$, the Hamiltonian $H(\mathbf{x}, \mathbf{v}) = \frac{1}{2}v^2 + \Phi$ is an integral of motion. If a potential $\Phi(R, z, t)$ is axisymmetric about the $z$ axis, the $z$-component of the angular momentum is an integral, and in a spherical potential $\Phi(r, t)$ the three components of the angular-momentum vector $\mathbf{L} = \mathbf{x} \times \mathbf{v}$ constitute three integrals of motion. However, we shall find in §3.2 that even when integrals exist, analytic expressions for them are often not available.

These concepts and their significance for the geometry of orbits in phase space are nicely illustrated by the example of motion in a spherically symmetric potential. In this case the Hamiltonian $H$ and the three components of the angular momentum per unit mass $\mathbf{L} = \mathbf{x} \times \mathbf{v}$ constitute four integrals. However, we shall find it more convenient to use $|\mathbf{L}|$ and the two independent components of the unit vector $\hat{\mathbf{n}} = \mathbf{L}/|\mathbf{L}|$ as integrals in place of $\mathbf{L}$. We have seen that $\hat{\mathbf{n}}$ defines the orbital plane within which the position vector $\mathbf{r}$ and the velocity vector $\mathbf{v}$ must lie. Hence we conclude that the two independent components of $\hat{\mathbf{n}}$ restrict the star's phase point to a four-dimensional region of phase space. Furthermore, the equations $H(\mathbf{x}, \mathbf{v}) = E$ and $|\mathbf{L}(\mathbf{x}, \mathbf{v})| = L$, where $L$ is a constant, restrict the phase point to that two-dimensional surface in this four-dimensional region on which $v_r = \pm\sqrt{2[E - \Phi(r)] - L^2/r^2}$ and $v_\psi = L/r$. In §3.5.1 we shall see that this surface is a torus and that the sign ambiguity in $v_r$ is analogous to the sign ambiguity in the $z$-coordinate of a point on the sphere $r^2 = 1$ when one specifies the point through its $x$ and $y$ coordinates. Thus, given $E$, $L$, and $\hat{\mathbf{n}}$, the star's position and velocity (up to its sign) can be specified by two quantities, for example $r$ and $\psi$.

Is there a fifth integral of motion in a spherical potential? To study this question, we examine motion in the potential

$$\Phi(r) = -GM\left(\frac{1}{r} + \frac{a}{r^2}\right). \tag{3.57}$$

For this potential, equation (3.11b) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + \left(1 - \frac{2GMa}{L^2}\right)u = \frac{GM}{L^2}, \tag{3.58}$$

the general solution of which is

$$u = C\cos\left(\frac{\psi - \psi_0}{K}\right) + \frac{GMK^2}{L^2}, \tag{3.59a}$$

where

$$K \equiv \left(1 - \frac{2GMa}{L^2}\right)^{-1/2}. \tag{3.59b}$$

Hence

$$\psi_0 = \psi - K\,\mathrm{Arccos}\left[\frac{1}{C}\left(\frac{1}{r} - \frac{GMK^2}{L^2}\right)\right], \tag{3.60}$$

where $t = \mathrm{Arccos}\,x$ is the multiple-valued solution of $x = \cos t$, and $C$ can be expressed in terms of $E$ and $L$ by

$$E = \tfrac{1}{2}\frac{C^2L^2}{K^2} - \tfrac{1}{2}\left(\frac{GMK}{L}\right)^2. \tag{3.61}$$

If in equations (3.59b), (3.60) and (3.61) we replace $E$ by $H(\mathbf{x},\mathbf{v})$ and $L$ by $|\mathbf{L}(\mathbf{x},\mathbf{v})| = |\mathbf{x}\times\mathbf{v}|$, the quantity $\psi_0$ becomes a function of the phase-space coordinates which is constant as the particle moves along its orbit. Hence $\psi_0$ is a fifth integral of motion. (Since the function $\mathrm{Arccos}\,x$ is multiple-valued, a judicious choice of solution is necessary to avoid discontinuous jumps in $\psi_0$.) Now suppose that we know the numerical values of $E$, $L$, $\psi_0$, and the radial coordinate $r$. Since we have four numbers—three integrals and one coordinate—it is natural to ask how we might use these numbers to determine the azimuthal coordinate $\psi$. We rewrite equation (3.60) in the form

$$\psi = \psi_0 \pm K\cos^{-1}\left[\frac{1}{C}\left(\frac{1}{r} - \frac{GMK^2}{L^2}\right)\right] + 2nK\pi, \tag{3.62}$$

where $\cos^{-1}(x)$ is defined to be the value of $\text{Arccos}\,(x)$ that lies between 0 and $\pi$, and $n$ is an arbitrary integer. If $K$ is irrational—as nearly all real numbers are—then by a suitable choice of the integer $n$, we can make $\psi$ modulo $2\pi$ approximate any given number as closely as we please. Thus for any values of $E$ and $L$, and any value of $r$ between the pericenter and apocenter for the given $E$ and $L$, an orbit that is known to have a given value of the integral $\psi_0$ can have an azimuthal angle as close as we please to any number between 0 and $2\pi$.

On the other hand, if $K$ is rational these problems do not arise. The simplest and most important case is that of the Kepler potential, when $a = 0$ and $K = 1$. Equation (3.62) now becomes

$$\psi = \psi_0 \pm \cos^{-1}\left[\frac{1}{C}\left(\frac{1}{r} - \frac{GM}{L^2}\right)\right] + 2n\pi, \tag{3.63}$$

which yields only two values of $\psi$ modulo $2\pi$ for given $E$, $L$ and $r$.

These arguments can be restated geometrically. The phase space has six dimensions. The equation $H(\mathbf{x}, \mathbf{v}) = E$ confines the orbit to a five-dimensional subspace. The vector equation $\mathbf{L}(\mathbf{x}, \mathbf{v}) = \textit{constant}$ adds three further constraints, thereby restricting the orbit to a two-dimensional surface. Through the equation $\psi_0(\mathbf{x}, \mathbf{v}) = \textit{constant}$ the fifth integral confines the orbit to a one-dimensional curve on this surface. Figure 3.1 can be regarded as a projection of this curve. In the Kepler case $K = 1$, the curve closes on itself, and hence does not cover the two-dimensional surface $H = \textit{constant}$, $\mathbf{L} = \textit{constant}$. But when $K$ is irrational, the curve is endless and densely covers the surface of constant $H$ and $\mathbf{L}$.

We can make an even stronger statement. Consider any volume of phase space, of any shape or size. Then if $K$ is irrational, the fraction of the time that an orbit with given values of $H$ and $\mathbf{L}$ spends in that volume does not depend on the value that $\psi_0$ takes on this orbit.

Integrals like $\psi_0$ for irrational $K$ that do not affect the phase-space distribution of an orbit, are called **non-isolating integrals**. All other integrals are called **isolating integrals**. The examples of isolating integrals that we have encountered so far, namely, $H$, $\mathbf{L}$, and the function $\psi_0$ when $K = 1$, all confine stars to a five-dimensional region in phase space. However, there can also be isolating integrals that restrict the orbit to a six-dimensional subspace of phase space—see §3.7.3. Isolating integrals are of great practical and theoretical importance, whereas non-isolating integrals are of essentially no value for galactic dynamics.